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A NONLINEAR MIXTURE THEORY FOR QUASI-ONE-DIMENSIONAL HEAT CONDUCTION IN FIBER REINFORCED COMPOSITES†

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Abstract—A nonlinear mixture theory with microstructure is constructed for quasi-one-dimensional heat conduction in fiber-reinforced composites. The nonlinearities considered result from temperature dependent thermal properties. In an attempt to infer the accuracy of the mixture theory, an initial boundary-value problem is selected and a finite difference solution of the mixture equations is compared with a finite element solution of the original equations. Excellent agreement is found between the two solutions for both global and micro-temperature fields. System sensitivity to nonlinearities is examined via numerical treatment of the mixture equations for Graphite/Epoxy and Carbon/Carbon composites. The results imply that variations of thermal properties with temperature can lead to significant differences between linear and nonlinear solutions.

INTRODUCTION

Heat conduction parallel to the fibers of a unidirectional fibrous composite is a problem of considerable practical importance for those cases in which such composites are used as thermal protection materials. Most materials currently used for this purpose exhibit temperature dependent diffusive properties. The resulting nonlinearities render analytical treatment intractable. On the other hand, if information regarding thermal states in the fiber and matrix is desired, then three spatial dimensions are involved and a direct numerical treatment may become extremely expensive.

A continuum mixture theory offers an alternative to a direct numerical approach when a slight loss in accuracy is acceptable; the latter usually lies within the zone of uncertainty of material properties for practical composites. Such a theory is developed in this paper using the Regular Asymptotic Method[1], and the concentric cylinders approximation[2].

Following development of the theory, numerical computations are performed for Graphite/Epoxy and Carbon/Carbon composites in an effort to assess the influence of nonlinearities. In an attempt to infer the accuracy of the nonlinear mixture theory, mixture solutions are also compared with the results obtained using a finite element solution of the original field equations.

FORMULATION

Consider a periodic hexagonal array of circular cylindrical fibers (Constituent 1) embedded in a matrix (Constituent 2) as shown in Fig. 1. With respect to a polar coordinate system $\bar{r}$, $\bar{\theta}$, $\bar{x}$, let the composite occupy the domain $0 \le \bar{x} \le l$, $0 \le \bar{r} < \infty$, $0 \le \bar{\theta} < 2\pi$. We assume that the initial conditions and temperature or heat flux boundary conditions on $\bar{x} = 0$, l are such that the temperature distribution is similar in each hexagonal cell. As a consequence, the external boundary of the cell becomes a surface of symmetry, and the components of the heat flux normal to this hexagonal boundary vanish. In addition, it is assumed that no thermal resistance exists across the interface between the fiber and the matrix.

The problem is further simplified by assuming, as in[3], that a typical hexagonal cell may be approximated by concentric cylinders. As a result of this approximation the temperature

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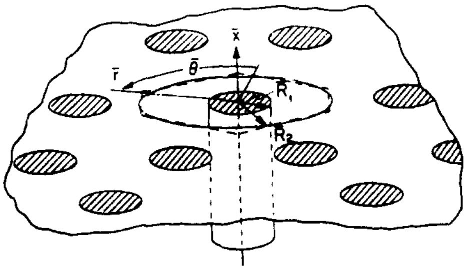

Fig. 1. Geometry and coordinate system.

distribution within the unit cell is axisymmetric, with zero heat flux normal to the outer circular boundary of the matrix.

The basic equations for the temperature fields $\bar T^{(\alpha)}$ and heat flux vector $(\bar Q_x^{(\alpha)}, \bar Q_r^{(\alpha)})$ are

(a) Conservation of energy:

$$\partial_{\bar x}(\bar Q_x^{(\alpha)}) + \frac{1}{\bar r}\partial_{\bar r}(\bar r \bar Q_r^{(\alpha)}) = -\bar\mu^{(\alpha)}(\bar T^{(\alpha)})\partial_{\bar t}\bar T^{(\alpha)} \text{ on } \bar A^{(\alpha)}, \tag{1}$$

where

$$\bar A^{(1)} \equiv [0, \bar R_1],\ \bar A^{(2)} \equiv [\bar R_1, \bar R_2];$$

(b) Fourier's law of heat conduction:

$$\bar Q_x^{(\alpha)} = -\bar k_{xx}^{(\alpha)}(\bar T^{(\alpha)})\partial_{\bar x}\bar T^{(\alpha)},\ \bar Q_r^{(\alpha)} = -\bar k_{rr}^{(\alpha)}(\bar T^{(\alpha)})\partial_{\bar r}\bar T^{(\alpha)} \text{ on } \bar A^{(\alpha)}. \tag{2}$$

In the above, superscript $\alpha = 1, 2$ refers to the fiber and matrix, respectively. The quantities $\bar\mu$, $\bar k_{xx}$, $\bar k_{rr}$ denote heat capacity, and axial- and transverse thermal conductivities. In what follows these are assumed to be polynomial functions in $\bar T^{(\alpha)}$. We use the notation $\partial_{\bar x}(\ ) \equiv \partial(\ )/\partial\bar x$, $\partial_{\bar r}(\ ) \equiv \partial(\ )/\partial\bar r$, and $\partial_{\bar t}(\ ) \equiv \partial(\ )/\partial\bar t$ where $\bar t$ denotes time.

In addition to (1) and (2), the initial boundary value problem is completed with

(c) Symmetry conditions:

$$\bar Q_r^{(2)}(\bar x, \bar R_2, \bar t) = 0; \tag{3}$$

(d) Interface conditions:

$$\bar T^{(1)} = \bar T^{(2)},\ \bar Q_r^{(1)} = \bar Q_r^{(2)} \text{ at } \bar r = \bar R_1; \tag{4}$$

(e) Initial condition at $\bar t = 0$ and appropriate boundary data at $\bar x = 0, l$.

Scaling procedure

Equations (1)–(4) and conditions (e) specify a well-posed problem involving $\bar x$, $\bar r$ and $\bar t$. The objective of the subsequent analysis is to derive simplified differential equations governing the macroscopic diffusion process which involve only one spatial dimension $\bar x$ and yet reflect the effect of conduction on the microscale.

Let $\bar\Lambda$ and $\bar R_2$ be a typical "macrodimension" and "microdimension," respectively. The macrolength $\bar\Lambda$ may be defined in terms of a characteristic thermal diffusion time in the longitudinal direction according to

$$\bar t_{(\Lambda)} \equiv \bar\mu_{(m)}\bar\Lambda^2/\bar k_{(m)} \tag{5a}$$

where $\bar\mu_{(m)}$, $\bar k_{(m)}$ denote *mixture* heat capacity and thermal conductivity, respectively. These quantities will be defined later. In addition to (5a), it is helpful to introduce a mixture heat flux $\bar Q_{(m)}$, based upon a reference temperature $\bar T_0$, according to

$$\bar Q_{(m)} = \bar k_{(m)}\bar T_0/\bar\Lambda, \tag{5b}$$

and a parameter ϵ as

$$\epsilon \equiv \bar{R}_2/\bar{\Lambda} \tag{5c}$$

which represents the ratio of micro-to-macro dimensions of the composite.

With the aid of the foregoing definitions, the following nondimensional variables are now defined:

$$\begin{aligned}
&(x, \epsilon r) = (\bar{x}, \bar{r})/\bar{\Lambda},\ R_1 = \bar{R}_1/\bar{R}_2,\ t = \bar{t}/\bar{t}_{(\Lambda)},\\
&(Q_x^{(\alpha)}, \epsilon Q_r^{(\alpha)}) = (\bar{Q}_x^{(\alpha)}, \bar{Q}_r^{(\alpha)})/\bar{Q}_{(m)},\\
&T^{(\alpha)} = (\bar{T}^{(\alpha)} - \bar{T}_0)/\bar{T}_0,\ \mu^{(\alpha)} = \bar{\mu}^{(\alpha)}/\bar{\mu}_{(m)},\\
&(k_{xx}^{(\alpha)}, k_{rr}^{(\alpha)}) = (\bar{k}_{xx}^{(\alpha)}, \bar{k}_{rr}^{(\alpha)})/\bar{k}_{(m)}.
\end{aligned} \tag{5d}$$

Using (5d), the basic equations can be rewritten as follows:

(a) Conservation of energy:

$$\partial_x Q_x^{(\alpha)} + \frac{1}{r}\partial_r(rQ_r^{(\alpha)}) = -\mu^{(\alpha)}\partial_t T^{(\alpha)} \text{ on } A^{(\alpha)}. \tag{6}$$

where

$$A^{(1)} \equiv [0, \sqrt{n^{(1)}}],\ A^{(2)} \equiv [\sqrt{n^{(1)}}, 1],$$

and $n^{(1)} = (\bar{R}_1/\bar{R}_2)^2$ is the volume fraction of the fiber in a cell;

(b) Fourier's law of heat conduction:

$$Q_x^{(\alpha)} = -k_{xx}^{(\alpha)}(T^{(\alpha)})\partial_x T^{(\alpha)} \text{ on } A^{(\alpha)} \tag{7a}$$

$$\epsilon^2 Q_r^{(\alpha)} = -k_{rr}^{(\alpha)}(T^{(\alpha)})\partial_r T^{(\alpha)} \text{ on } A^{(\alpha)}; \tag{7b}$$

(c) Symmetry condition:

$$Q_r^{(2)}(x, 1, t) = 0; \tag{8}$$

(d) Interface conditions:

$$T^{(1)}(x, \sqrt{n^{(1)}}, t) = = T^{(2)}(x, \sqrt{n^{(1)}}, t), \tag{9a}$$

$$Q_r^{(1)}(x, \sqrt{n^{(1)}}, t) = Q_r^{(2)}(x, \sqrt{n^{(1)}}, t); \tag{9b}$$

(e) Initial condition at $t = 0$, and appropriate boundary data at $x = 0, l/\bar{\Lambda}$.

In the above $A^{(\alpha)}$ denotes $\bar{A}^{(\alpha)}$ in nondimensional coordinates; partial derivatives are now defined by $\partial_x(\ ) \equiv \partial(\ )/\partial x$, $\partial_r(\ ) \equiv \partial(\ )/\partial r$ and $\partial_t(\ ) \equiv \partial(\ )/\partial t$.

Mixture equations

Mixture equations for the diffusion process are obtained by averaging (6) over the cross-sectional area $A^{(\alpha)}$. For this purpose, we define averaged quantities such that

$$f^{(1a)}(x, t) \equiv \frac{1}{n^{(1)}}\int_0^{\sqrt{n^{(1)}}} 2rf^{(1)}(x, r, t)\,\mathrm{d}r, \tag{10a}$$

$$f^{(2a)}(x, t) \equiv \frac{1}{n^{(2)}}\int_{\sqrt{n^{(1)}}}^{1} 2rf^{(2)}(x, r, t)\,\mathrm{d}r, \tag{10b}$$

where $n^{(1)}$ and $n^{(2)}$ are volume fractions of the two constituents, such that

$$n^{(1)} + n^{(2)} = 1. \tag{11}$$

Upon integration of (6) according to (10), and with use of the interface condition (9b), we obtain the standard binary mixture form[3],

$$\begin{aligned}\partial_x Q_x^{(1p)} + \mu^{(1p)}\partial_t T^{(1a)} &= -P,\\ \partial_x Q_x^{(2p)} + \mu^{(2p)}\partial_t T^{(2a)} &= P.\end{aligned} \tag{12}$$

In the above equations

$$Q_x^{(\alpha p)} = n^{(\alpha)} Q_x^{(\alpha a)},\ \mu^{(\alpha p)} = n^{(\alpha)}\mu^{(\alpha)}(T^{(\alpha a)}), \tag{13}$$

are "partial" heat fluxes and heat capacities, and

$$P \equiv 2\sqrt{n^{(1)}}\, Q_r^{(1)}(x, \sqrt{n^{(1)}}, t) = 2\sqrt{n^{(1)}}\, Q_r^{(2)}(x, \sqrt{n^{(1)}}, t) \tag{14}$$

is an "interaction" term reflecting heat transfer between the fiber and the matrix across the interface. At this stage the mixture eqns (12) are exact in the average sense. The primary object of the subsequent analysis is to construct a *closed* theory by determining the functional dependence of the partial heat flux $Q_x^{(\alpha p)}$ and the interaction term P on $T^{(1a)}$ and $T^{(2a)}$.

Asymptotic expansions

A fundamental premise is now introduced; we assume that the ratio of micro-to-macro dimensions is small compared to unity, i.e.

$$\epsilon = \bar{R}_2/\bar{\Lambda} \ll 1. \tag{15}$$

The assumption (15) suggests the following regular asymptotic expansion for all dependent variables, denoted by $G^{(\alpha)}$:

$$G^{(\alpha)}(x, r, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{2n} G_{(2n)}^{(\alpha)}(x, r, t). \tag{16}$$

If (16) is substituted into the governing equations (6) and (7) and the coefficients of similar order of ϵ are equated, one obtains a system of equations for each $n = 0, 1, 2, \ldots$. Recall that we have assumed all thermal conductivities and heat capacities may be represented by polynomial functions of temperature, i.e.

$$k_{xx}^{(\alpha)}(T^{(\alpha)}) = k_{xx(0)}^{(\alpha)}(T_{(0)}^{(\alpha)}) + \epsilon^2 k_{xx(2)}^{(\alpha)}(T_{(0)}^{(\alpha)}, T_{(2)}^{(\alpha)}) + O(\epsilon^4), \tag{17a}$$

$$k_{rr}^{(\alpha)}(T^{(\alpha)}) = k_{rr(0)}^{(\alpha)}(T_{(0)}^{(\alpha)}) + \epsilon^2 k_{rr(2)}^{(\alpha)}(T_{(0)}^{(\alpha)}, T_{(2)}^{(\alpha)}) + O(\epsilon^4), \tag{17b}$$

$$\mu^{(\alpha)}(T^{(\alpha)}) = \mu_{(0)}^{(\alpha)}(T_{(0)}^{(\alpha)}) + \epsilon^2 \mu_{(2)}^{(\alpha)}(T_{(0)}^{(\alpha)}, T_{(2)}^{(\alpha)}) + O(\epsilon^4). \tag{17c}$$

The lowest order system is

$$\partial_x Q_{x(0)}^{(\alpha)} + \frac{1}{r}\partial_r(rQ_{r(0)}^{(\alpha)}) = -\mu_{(0)}^{(\alpha)}(T_{(0)}^{(\alpha)})\partial_t T_{(0)}^{(\alpha)} \text{ on } A^{(\alpha)}, \tag{18a}$$

$$Q_{x(0)}^{(\alpha)} = -k_{xx(0)}^{(\alpha)}(T_{(0)}^{(\alpha)})\partial_x T_{(0)}^{(\alpha)} \text{ on } A^{(\alpha)}, \tag{18b}$$

$$-k_{rr(0)}^{(\alpha)}(T_{(0)}^{(\alpha)})\partial_r T_{(0)}^{(\alpha)} = 0 \text{ on } A^{(\alpha)}. \tag{18c}$$

Equation (18c) implies that

$$T_{(0)}^{(\alpha)} = T_{(0)}^{(\alpha)}(x, t). \tag{19}$$

We now develop a mixture theory based upon the lowest order system.

MIXTURE THEORY BASED ON LOWEST ORDER SYSTEM

We obtain mixture constitutive relations for $Q_x^{(\alpha p)}$ and P from the lowest order system (18). With (19), eqn (18b) may be averaged according to (10) which furnishes, to lowest order accuracy,

$$Q_x^{(\alpha p)} = -n^{(\alpha)} k_{xx}^{(\alpha)}(T^{(\alpha a)})\partial_x T^{(\alpha a)} \tag{20}$$

(note that $k_{xx(0)}^{(\alpha)}(T^{(\alpha a)}) = k_{xx}^{(\alpha)}(T^{(\alpha a)})$ due to the polynomial dependency of $k_{xx}^{(\alpha)}$ on $T^{(\alpha)}$). In order to obtain a relation between P and $T^{(\alpha a)}$, it is necessary to consider $Q_{r(0)}^{(\alpha)}$, $T_{(2)}^{(\alpha)}$, and to satisfy the continuity of temperature (9a) including $O(\epsilon^2)$ terms.

To begin, one finds from (7b), (16) and (17b) that

$$Q_{r(0)}^{(\alpha)} = -k_{rr}^{(\alpha)}(T_{(0)}^{(\alpha)})\partial_r T_{(2)}^{(\alpha)}, \tag{21}$$

where $k_{rr(0)}^{(\alpha)}(T_{(0)}^{(\alpha)}) = k_{rr}^{(\alpha)}(T_{(0)}^{(\alpha)})$ has been used. Next, using (19) and (21), eqn (18a) becomes

$$\frac{1}{r}\partial_r\{rk_{rr}^{(\alpha)}(T_{(0)}^{(\alpha)})\partial_r T_{(2)}^{(\alpha)}\} = \varphi^{(\alpha)}(x, t). \tag{22}$$

The function $\varphi^{(\alpha)}(x, t)$ can be related to the interaction term P, defined by (14), by integration of (22) over $A^{(\alpha)}$. With the use of the symmetry condition (8), we find

$$n^{(1)}\varphi^{(1)} = -P,\ n^{(2)}\varphi^{(2)} = P. \tag{23}$$

Thus, to $O(\epsilon^2)$ one obtains the following boundary value problem for the function $T_{(2)}^{(\alpha)}$:

$$\frac{1}{r}\partial_r\{rk_{rr}^{(1)}(T_{(0)}^{(1)}\partial_r T_{(2)}^{(1)}\} = -\frac{P}{n^{(1)}} \text{ on } A^{(1)}, \tag{24a}$$

$$\frac{1}{r}\partial_r\{rk_{rr}^{(2)}(T_{(0)}^{(2)})\partial_r T_{(2)}^{(2)}\} = \frac{P}{n^{(2)}} \text{ on } A^{(2)}, \tag{24b}$$

$$\partial_r T_{(2)}^{(2)} = 0 \text{ on } r = 1. \tag{25a}$$

$$T_{(0)}^{(1)}(x, t) + \epsilon^2 T_{(2)}^{(1)} = T_{(0)}^{(2)}(x, t) + \epsilon^2 T_{(2)}^{(2)} \text{ on } r = \sqrt{n^{(1)}}, \tag{25b}$$

$$k_{rr}^{(1)}(T_{(0)}^{(1)})\partial_r T_{(2)}^{(1)} = k_{rr}^{(2)}(T_{(0)}^{(2)})\partial_r T_{(2)}^{(2)} \text{ on } r = \sqrt{n^{(1)}}. \tag{25c}$$

Because of the Neumann condition (25a), the solution to the above problem is unique within a function $H(x, t)$. Without any loss of generality we may combine this function with $T_{(0)}^{(\alpha)}(x, t)$ such that:

$$\tilde{T}_{(0)}^{(\alpha)}(x, t) \equiv T_{(0)}^{(\alpha)}(x, t) + \epsilon^2 H(x, t). \tag{26}$$

Using (26), uniqueness for the field $T_{(2)}^{(\alpha)}$ can be obtained by setting

$$T_{(2)}^{(1)} = 0 \quad \text{on} \quad r = 0. \tag{27}$$

Since the jump condition (25b) involving $T_{(2)}^{(\alpha)}$ on $r = \sqrt{n^{(1)}}$ is independent of r, the problem can be rewritten as a continuous field problem by introducing new variables $T^{*(\alpha)}$ such that

$$PT^{*(1)} \equiv T_{(2)}^{(1)};\ PT^{*(2)} \equiv T_{(2)}^{(2)} + \frac{1}{\epsilon^2}(\tilde{T}_{(0)}^{(2)} - \tilde{T}_{(0)}^{(1)}). \tag{28}$$

The boundary value problem in terms of the new dependent variables $T^{*(\alpha)}$ is

$$\frac{1}{r}\partial_r\{rk_{rr}^{(1)}(T_{(0)}^{(1)})\partial_r T^{*(1)}\} = -\frac{1}{n^{(1)}} \text{ on } A^{(1)}, \tag{29a}$$

$$\frac{1}{r}\partial_r\{rk_{rr}^{(2)}(T_{(0)}^{(2)})\partial_r T^{*(2)}\} = \frac{1}{n^{(2)}} \text{ on } A^{(2)}, \tag{29b}$$

$$\partial_r T^{*(2)} = 0 \quad \text{on } r = 1, \tag{30}$$

$$T^{*(1)} = T^{*(2)},\ k_{rr}^{(1)}(T_{(0)}^{(1)})\partial_r T^{*(1)} = k_{rr}^{(2)}(T_{(0)}^{(2)})\partial_r T^{*(2)} \quad \text{on } r = \sqrt{n^{(1)}}, \tag{31}$$

$$T^{*(1)} = 0 \quad \text{on } r = 0. \tag{32}$$

After slight manipulation, $T^{*(1)}$ and $T^{*(2)}$ are found to be

$$T^{*(1)}(r) = -\frac{r^2}{4n^{(1)}k^{(1)}_{(rr)}(T^{(1)}_{(0)})}. \tag{33a}$$

$$T^{*(2)}(r) = \frac{1}{4n^{(2)}k^{(2)}_{rr}T^{(2)}_{(0)})}\left(r^2 - n^{(1)} + \ln\frac{n^{(1)}}{r^2}\right) - \frac{1}{4k^{(1)}_{rr}(T^{(1)}_{(0)})}. \tag{33b}$$

Using (26) and (28), the temperature fields can be written to $O(\epsilon^2)$ accuracy as

$$\begin{aligned} T^{(1)} &= \bar{T}^{(1)}_{(0)}(x, t) + \epsilon^2 PT^{*(1)}(x, r, t) + O(\epsilon^4), \\ T^{(2)} &= \bar{T}^{(1)}_{(0)}(x, t) + \epsilon^2 PT^{*(2)}(x, r, t) + O(\epsilon^4). \end{aligned} \tag{34}$$

Upon averaging (34) according to (10) one obtains

$$T^{(\alpha a)}(x, t) = \bar{T}^{(1)}_{(0)}(x, t) + \epsilon^2 PT^{*(\alpha a)}(x, t) \tag{35}$$

Eliminating $\bar{T}^{(1)}_{(0)}$ from the above, the interaction term can be written as a function of the averaged temperature $T^{(\alpha a)}(x, t)$. Thus,

$$P = \zeta(T^{(1a)} - T^{(2a)})/\epsilon^2 \tag{36}$$

where

$$\zeta \equiv (T^{*(1a)} - T^{*(2a)})^{-1}. \tag{37}$$

By substituting the averages of (33) into (37) one finds

$$\zeta(T^{(1)}_{(0)}, T^{(2)}_{(0)}) = 8 \Bigg/ \left\{\frac{1}{k^{(1)}_{rr}(T^{(1)}_{(0)})} - \frac{1}{k^{(2)}_{rr}(T^{(2)}_{(0)})}\left(1 + \frac{2}{n^{(2)}} + \frac{\ln n^{(1)^2}}{n^{(2)^2}}\right)\right\}. \tag{38}$$

Approximating $T^{(\alpha)}_{(0)}$ in (38) by $T^{(\alpha a)}$, the foregoing analysis completes the construction of a binary mixture theory of the composite. The basic equations of the theory are:

$$\partial_x Q_x^{(1p)} + \mu^{(1p)}(T^{(1a)})\partial_t T^{(1a)} = -P, \tag{12a}$$

$$\partial_x Q_x^{(2p)} + \mu^{(2p)}(T^{(2a)})\partial_t T^{(2a)} = P, \tag{12b}$$

$$Q_x^{(\alpha p)} = -n^{(\alpha)}k^{(\alpha)}_{xx}(T^{(\alpha a)})\partial_x T^{(\alpha a)};\ \alpha = 1, 2, \tag{20}$$

$$P = \zeta(T^{(1a)} - T^{(2a)})/\epsilon^2, \tag{36}$$

$$\zeta(T^{(1a)}, T^{(2a)}) = 8 \Bigg/ \left\{\frac{1}{k^{(1)}_{rr}(T^{(1a)})} - \frac{1}{k^{(2)}_{rr}(T^{(2a)})}\left(1 + \frac{2}{n^{(2)}} + \frac{\ln n^{(1)^2}}{n^{(2)^2}}\right)\right\}. \tag{38}$$

TEMPERATURE MICROSTRUCTURE AND MIXTURE QUANTITIES

Solution of the foregoing set of equations, with appropriate boundary and initial data, supplies the macroscopic temperature fields. One may also obtain the temperature microstructure to a specified degree of accuracy. To this end one first calculates

$$\bar{T}^{(1)}_{(0)}(x, t) = T^{(\alpha a)}(x, t) - \zeta(T^{(1a)} - T^{(2a)})T^{*(\alpha a)};\ \alpha = 1 \text{ or } 2, \tag{39}$$

from (35) and (36), and substitutes the result in (34).

To obtain expressions for the mixture conductivity and specific heat to be used in (5), we observe that eqns (12), (20) and (36) yield

$$\begin{aligned} \partial_x\{n^{(1)}k^{(1)}_{xx}(T^{(1a)})\partial_x T^{(1a)}\} - n^{(1)}\mu^{(1)}(T^{(1a)})\partial_t T^{(1a)} &= \zeta(T^{(1a)}, T^{(2a)})\left(\frac{T^{(1a)} - T^{(2a)}}{\epsilon^2}\right), \\ \partial_x\{n^{(2)}k^{(2)}_{xx}(T^{(2a)})\partial_x T^{(2a)}\} - n^{(2)}\mu^{(2)}(T^{(2a)})\partial_t T^{(2a)} &= -\zeta(T^{(1a)}, T^{(2a)})\left(\frac{T^{(1a)} - T^{(2a)}}{\epsilon^2}\right). \end{aligned} \tag{40}$$

The addition of (40) gives

$$\partial_x\{n^{(1)}k_{xx}^{(1)}(T^{(1a)})\partial_x T^{(1a)}+n^{(2)}k_{xx}^{(2)}(T^{(2a)})\partial_x T^{(2a)}\}-\{n^{(1)}\mu^{(1)}(T^{(1a)})\partial_t T^{(1a)}+n^{(2)}\mu^{(2)}(T^{(2a)})\partial_t T^{(2a)}\}=0. \tag{41}$$

In the limit as $\epsilon\rightarrow 0$, the mixture theory reduces to an elementary theory for heat conduction in a homogeneous material,

$$\underset{\epsilon\to 0}{\text{limit}}\; T^{(1a)}=T^{(2a)}. \tag{42}$$

For materials with constant $k_{xx}^{(\alpha)}$ and $\mu^{(\alpha)}$ eqn (41) furnishes the appropriate mixture quantities:

$$\bar{k}_{(m)}=n^{(1)}\bar{k}_{xx}^{(1)}+n^{(2)}\bar{k}_{xx}^{(2)},\quad \bar{\mu}_{(m)}=n^{(1)}\bar{\mu}^{(1)}+n^{(2)}\bar{\mu}^{(2)}. \tag{43}$$

Application of an analogous procedure for materials with temperature dependent thermal properties requires reasonable estimates for the upper and lower bounds $\bar{T}_M$ and $\bar{T}_m$ of $\bar{T}^{(\alpha a)}(x,t)$; $\alpha=1,2$ of the specific problem under consideration. Accordingly, it is appropriate to define the reference temperature $\bar{T}_0$ in terms of $\bar{T}_M$ and $\bar{T}_m$ as

$$\bar{T}_0\equiv(\bar{T}_M+\bar{T}_m)/2. \tag{44}$$

The mixture quantities can then be defined as:

$$\bar{k}_{(m)}=\frac{1}{\bar{T}_M-\bar{T}_m}\int_{\bar{T}_m}^{\bar{T}_M}\{n^{(1)}\bar{k}_{xx}^{(1)}(\xi)+n^{(2)}\bar{k}_{xx}^{(2)}(\xi)\}\,\mathrm{d}\xi, \tag{45}$$

$$\bar{\mu}_{(m)}=\frac{1}{\bar{T}_M-\bar{T}_m}\int_{\bar{T}_m}^{\bar{T}_M}\{n^{(1)}\bar{\mu}^{(1)}(\xi)+n^{(2)}\bar{\mu}^{(2)}(\xi)\}\,\mathrm{d}\xi. \tag{46}$$

It is noted that the above definitions are not unique. Thus eqns (45) and (46) provide only one of many possible choices of appropriate mixture quantities. To assure the validity of the asymptotic expansion of (6) and (7), however, it is necessary that all material properties be of $O(1)$.

NUMERICAL RESULTS

In order to test the accuracy of the nonlinear mixture theory, calculations have been performed to determine the evolution of the temperature field in a quiescent half-space $\bar{x}\leq 0$, subject to the boundary condition

$$\bar{T}(0,\bar{r},\bar{t})=\begin{cases}200, & 0\leq\bar{t}\leq\bar{t}_B\\ 0, & \bar{t}_B<\bar{t}\end{cases}. \tag{47}$$

This boundary value problem has been solved by using (i) a finite difference solution of the mixture eqns (12), and (ii) a finite element solution of (1)–(4). The finite element solution of (1)–(4) is referenced as the "exact" solution, and is used as the correlating norm. The material properties used for a first set of computations were selected to represent a Graphite/Epoxy fiber-reinforced composite [5] and are given in Table 1. For the corresponding linear analysis the thermal properties at $\bar{T}^{(\alpha)}=0$ were selected. Calculations have been carried out for $\epsilon=0.9$, thereby comprising a severe test of the mixture theory. Since there is no intrinsic axial length scale l in the half-plane problem, the pulse duration $\bar{t}_s$ has been used for scaling in (5). Small values of ϵ correspond, therefore, to short pulse lengths ($\bar{t}_s$).

The average temperature profiles in the two constituents, a short time before and after the termination of boundary heating, are shown in Figs. 2(a) and (b). In these figures the "linear" analysis refers to the mixture theory for materials with temperature independent properties. (This solution has been shown to be quite accurate in [3].) In Fig. 3 the evolution of average temperatures is depicted at a given distance from the boundary.

From these figures it is observed that the agreement between the mixture theory prediction

Table 1. Material properties used for computation

Volume Fractions: $n^{(1)} = 0.2$ (fiber)
$n^{(2)} = 0.8$ (matrix)

Thermal Properties:

	Axial Thermal Conductivity $\bar{k}_{xx}^{(\alpha)}$ Radial Thermal Conductivity $\bar{k}_{rr}^{(\alpha)}$	Heat Capacity $\bar{\mu}^{(\alpha)}$
Fiber (α=1)	$\bar{k}_{xx}^{(1)} = 10 - 0.01\ \bar{T}^{(1)}$ $\bar{k}_{rr}^{(1)} = 8 - 0.008\ \bar{T}^{(1)}$	$\bar{\mu}^{(1)} = 0.875 + 0.004\ \bar{T}^{(1)}$ $- 5 \times 10^{-6}\ (\bar{T}^{(1)})^2$
Matrix (α=2)	$\bar{k}_{xx}^{(2)} = 1 + 0.005\ \bar{T}^{(2)}$ $\bar{k}_{rr}^{(2)} = 1 + 0.005\ \bar{T}^{(2)}$	$\bar{\mu}^{(2)} = 1 + 0.002\ \bar{T}^{(2)}$ $+ 1.5 \times 10^{-6}\ (\bar{T}^{(2)})^2$

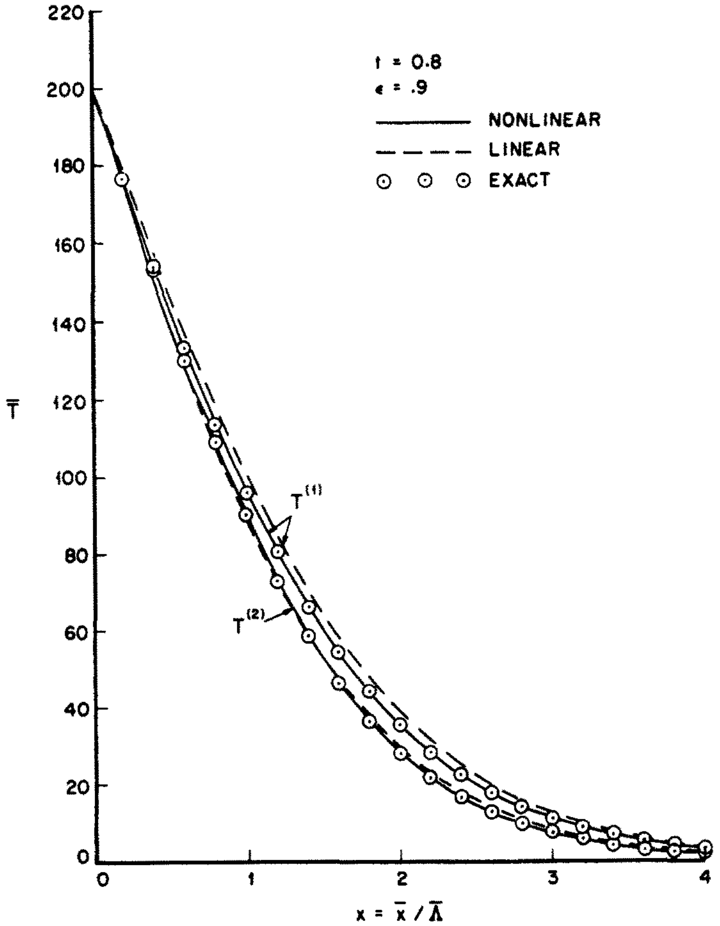

Fig. 2(a). Profiles of average temperatures at $t = 0.8$ ($\epsilon = 0.9$).

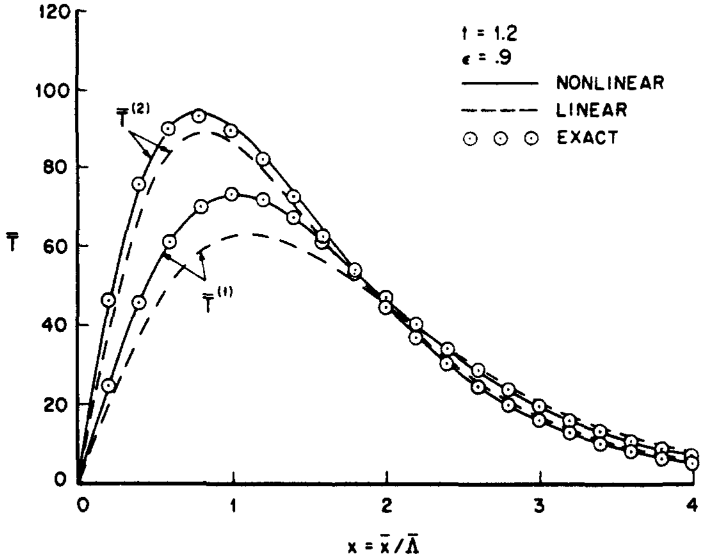

Fig. 2(b). Profile of average temperature at $t = 1.2$ ($\epsilon = 0.9$).

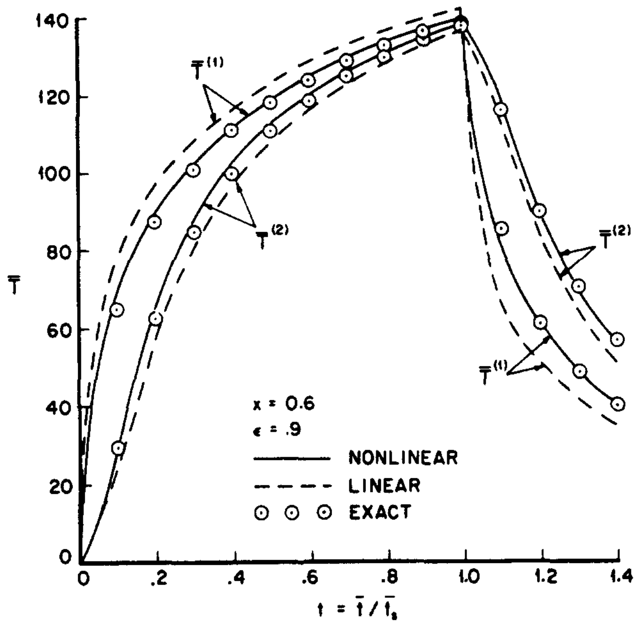

Fig. 3. Evolution of average temperatures at $x = 0.6$ ($\epsilon = 0.9$).

and the exact solution is excellent even in the situation $\epsilon \approx 1$. The significant differences evident between the two averaged temperatures endorse the soundness of the present approach when compared to an effective modulus-type theory. Such a theory allows only one macroscopic temperature field. An additional computation revealed that the effect of transverse isotropy was negligible for the material properties chosen in Table 1.

The ability of the mixture theory to predict the temperature microstructure is illustrated in Fig. 4. The radial distribution of temperature obtained from the mixture theory is practically indistinguishable from the "exact" solution for this problem.

In the foregoing problem the thermal properties depend weakly on temperature; the results imply, however, that the temperature dependency of thermal properties can significantly influence the results. In an effort to demonstrate this point, an initial-value problem similar to that of (47) was posed for a Carbon/Carbon-type composite. Here, the temperature on the

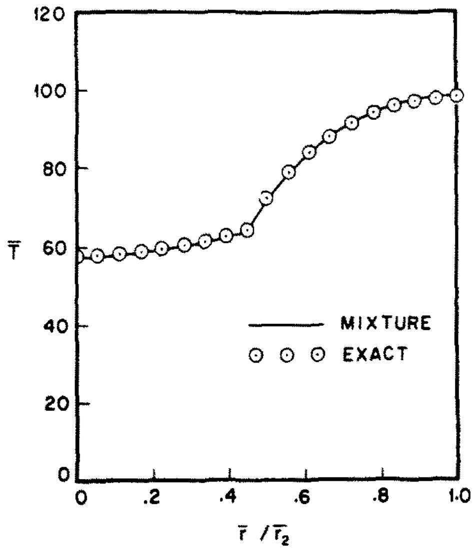

Fig. 4. Temperature nicrostructure at $x = 0.5$, $t = 1.2$ ($\epsilon = 0.9$).

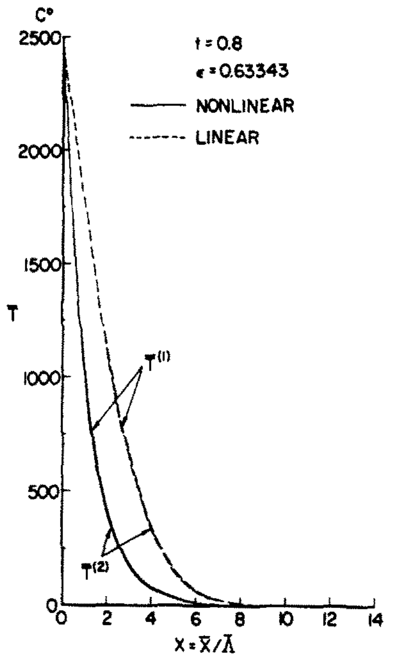

Fig. 5. Profiles of Average temperatures in carbon/carbon at $t = 0.8$, ($\epsilon = 0.63343$).

boundary was taken as

$$\bar{T}(0,\bar{r},\bar{t}) = \begin{cases} 2500°\mathrm{C} & 0 \leq \bar{t} \leq \bar{t}_s \\ 0°\mathrm{C} & \bar{t}_s < \bar{t} \end{cases}. \tag{48}$$

The material properties assumed for the Carbon/Carbon constituents are listed in Table 2. The temperature dependency of thermal conductivity, and specific heat for a graphite fiber was inferred from the ATJ-S(WS) graphite data of [6]. For the amorphous carbon matrix, the thermal conductivity was determined by averaging the data given in [8]. The specific heat of the matrix was assumed to be identical to that of the fiber. Densities of both fiber and matrix were taken from [8].

In the numerical computation corresponding to (48), $\bar{t}_s$ was taken as 0.0087538, and as a result, $\epsilon = 0.63343$. Thus, while the temperature profiles depicted in Fig. 5–7 are unconfirmed by either experimental data or "exact" computation, ϵ is sufficiently small to ensure that results are reasonably representative of a Carbon/Carbon composite possessing the geometry and constituent properties of Table 2.

Figures 5–7 exhibit considerable differences between linear and nonlinear computations.

CONCLUDING REMARKS

A nonlinear mixture theory has been constructed for diffusion in a fiber-reinforced composite where the nonlinearity is induced by temperature dependent material properties. Without losing the essential details of the local temperature distributions, the theory yields macroscopic diffusion equations involving just one spatial variable. The computational efficiency of the mixture equations, as compared to a direct numerical approach of the original problem, is significant due to the nonlinearity involved. The excellent agreement between the mixture theory predictions and the exact solution of a particular boundary value problem indicates that

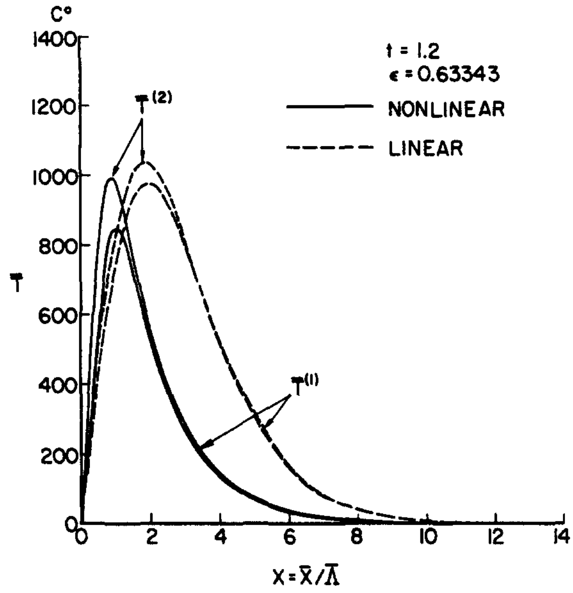

Fig. 6. Evolution of average temperatures in carbon/carbon at $x = 1.2$ ($\epsilon = 0.63343$).

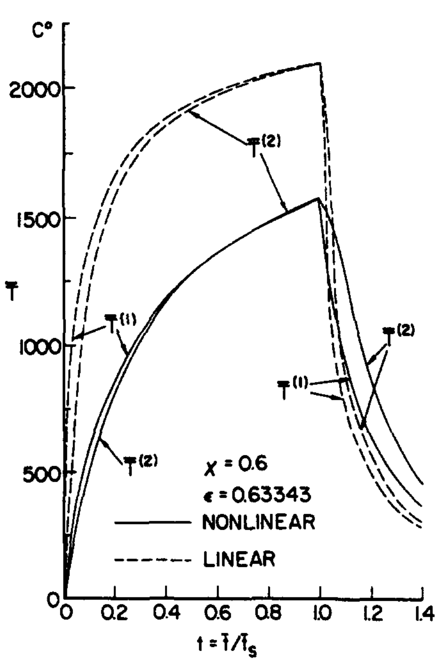

Fig. 7. Evolution of average temperatures in carbon/carbon at $x = 0.6$ ($\epsilon = 0.63343$).

Table 2. Material properties of carbon/carbon composite

Volume Fractions: $n^{(1)} = 0.2$, $n^{(2)} = 0.8$

Densities : $\rho^{(1)} = 1.767\ \mathrm{g/cm^3}$, $\rho^{(2)} = 1.67\ \mathrm{g/cm^3}$

Thermal Properties:

	$f^{(\alpha)} = \sum_{n=0}^{4} a_n \left(\frac{\bar{T}^{(\alpha)}}{1000}\right)^n$				
	a_0	a_1	a_2	a_3	a_4
$\bar{k}_{xx}^{(1)}$	1069.42	-1119.78	646.40	-179.327	19.1263
$\bar{k}_{xx}^{(2)}$	178.20	-31.678	6.015	0	0
$\bar{k}_{rr}^{(1)}$	110.84	-106.12	54.24	-13.52	1.1647
$\bar{k}_{rr}^{(2)}$	178.20	-31.678	6.015	0	0
$\bar{\mu}^{(1)}$	0.2937	1.0962	-0.8448	0.3129	-0.0436
$\bar{\mu}^{(2)}$	0.2776	1.0361	-0.7984	0.2958	-0.0412

(Note: $\bar{\mu}^{(\alpha)} = \bar{\rho}^{(\alpha)} \cdot \bar{C}_p^{(\alpha)}$) $[k_{xx}^{(\alpha)}] = [k_{rr}^{(\alpha)}] = \mathrm{cal/cm - hr - {}^\circ C}$

$[\bar{\mu}^{(\alpha)}] = \mathrm{cal/cm^3 \cdot {}^\circ C}$

the model proposed is quite effective. In addition, it appears likely that the theory may be extendable to a wide variety of other composite geometries of interest.

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